1. Metric Space

We begin with some preliminaries. A set is a collection of objects or elements. Examples:

- integers: \( \mathbb{N} = \{ \ldots, -1, 0, 1, \ldots \} \)
- rational numbers: \( \mathbb{Q} = \{ m/n \mid m, n \in \mathbb{N}, n \neq 0 \} \)
- real numbers: \( \mathbb{R} \)
- finite-dimensional real space: \( \mathbb{R}^n \)
- the set \( X = \{ x \in \mathbb{R}^2 \mid x = \alpha z, \alpha \in \mathbb{R} \} \) where \( z \in \mathbb{R}^2 \)
- the set of all infinite sequences \((x_0, x_1, x_2, \ldots)\), where \( x_i \in \mathbb{R} \) for all \( i \).
- the set of all real-valued polynomials defined on interval \([a, b]\), \( X = \{ x: [a, b] \to \mathbb{R} \mid x(t) = \sum_{i=0}^{m} \alpha_i t^i, t \in [a, b], \alpha_i \in \mathbb{R}, i = 1, \ldots, m, m \in \mathbb{N}_+ \} \).

Review set operations: empty set \( \emptyset \), \( A \cup B \), \( A \cap B \), \( A^c \) (complement), \( A \times B \), \( A \subset B \).

Let \( X \) and \( Y \) be two non-empty sets. \( f \) is called a function (mapping) from \( X \) into \( Y \) if for every \( x \in X \) there exists a unique \( y \in Y \) such that \( y = f(x) \), denote \( f: X \to Y \). The set \( X \) is called the domain of \( f \), and the set \( \mathcal{R}(f) = \{ y \in Y \mid \exists x \in X \ni f(x) = y \} \) is called the range of \( f \). The set \( \text{Gr}(f) = \{ (x, y) \in X \times Y \mid x \in X, y \in Y, f(x) = y \} \) is call the graph of \( f \).

Functions are further classified according to the properties of their range.

- **Onto** (surjective): \( \forall y \in Y, \exists \text{ at least one } x \in X, y = f(x) \).
- **One-to-one** (injective): \( \forall y \in Y, \exists \text{ at most one } x \in X, y = f(x) \).
- **One-to-one and onto** (bijective): \( \forall y \in Y, \exists \text{ unique } x \in X, y = f(x) \).

Note: Multivalued functions are not allowed. They are called correspondence.

Examples:

- \( f: \mathbb{R} \to \mathbb{R}_+, f(x) = x^2 \).
- \( f: \mathbb{N} \to \mathbb{N}, f(x) = 2x \).
- \( f: \mathbb{R} \to \mathbb{R}, f(x) = 2x \).
- \( f: \mathbb{Q} \to \mathbb{N} \times \mathbb{N}, f(q) = m/n \).
If $f$ is a one-to-one mapping from $X$ into $Y$, then $f$ has an inverse, $f^{-1} : \mathcal{R}(f) \to X$. $f^{-1}$ is onto. For all $x \in X$, $f^{-1}(f(x)) = x$.

1.1. Metric space

**Definition 1.1.** A metric space $\langle X, d \rangle$ is a nonempty set $X$ together with a real-valued function $d$ called a metric, $d : X \times X \to \mathbb{R}$, that satisfies the following properties: $\forall x, y, z \in X$,

(i) Nonnegativity: $d(x, y) \geq 0$, $d(x, y) = 0$ iff $x = y$.

(ii) Symmetry: $d(x, y) = d(y, x)$.

(iii) Triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)$.

Examples of metric space:

[1] The Real line $\langle \mathbb{R}, d \rangle$: for any $x, y \in \mathbb{R}$, $d(x, y) = |x - y|$.

[2] $\langle \mathbb{R}^n, d_1 \rangle$: for any $x, y \in \mathbb{R}^n$, 

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| \quad (1)$$

[3] Euclidean space $\langle \mathbb{R}^n, d_2 \rangle$: for any $x, y \in \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$,

$$d_2(x, y) = \left( \sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}. \quad (2)$$

Note: Cauchy-Schwarz inequality: for any $a_i, b_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$,

$$\left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right) \quad (3)$$

Use Cauchy-Schwarz inequality to prove the metric satisfies the triangle inequality.

[4] $\langle \mathbb{R}^n, d_\infty \rangle$: for any $x, y \in \mathbb{R}^n$,

$$d_\infty(x, y) = \max \{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\} \quad (4)$$

[5] Let $X$ be an arbitrary non-empty set. For any $x, y \in X$,

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} \quad (5)$$

$d$ is called discrete metric, and can be used to metrize any set.
\[ C[a, b], d_\infty \]: \( C[a, b] \) is the set of all real valued continuous functions defined on \([a, b]\). For any \( x, y \in C[a, b] \),

\[
d_\infty(x, y) = \sup_{t \in [a, b]} \{|x(t) - y(t)|\}
\]  

(6)

Note: Metric \( d \) is not unique. If \( d \) is a metric on \( X \), for any \( \alpha \in \mathbb{R}_+ \), \( \alpha d \) is a metric for \( X \).

**Definition 1.2.** Let \( \langle X, d \rangle \) be a metric space. If there exists a \( r \in \mathbb{R}, r > 0 \) such that \( \forall x, y \in X, d(x, y) \leq r \), \( \langle X, d \rangle \) is a bounded metric space. Otherwise, it is unbounded.

In above examples, metric space in [5] is bounded. All the rest are unbounded.

**Theorem 1.1.** Let \( X \) be a non-empty set, define \( d : X \times X \rightarrow \mathbb{R} \). Then \( d \) is a metric iff \( \forall x, y, z \in X, d(x, y) = 0 \) iff \( x = y \).

(i) \( d(x, y) = 0 \) iff \( x = y \).

(ii) \( d(x, y) \leq d(z, x) + d(z, y) \).

Let \( \langle X, d \rangle \) be a metric space, and let \( Y \) be a non-empty subset of \( X \). If \( d' \) denotes the restriction of \( d \) to \( Y \times Y \), i.e., if \( d'(x, y) = d(x, y), \forall x, y \in Y \), then \( \langle Y, d' \rangle \) is a metric space. That is, every subset of a metric space is a metric space. Call \( d' \) the metric induced by \( d \) on \( Y \). \( \langle Y, d' \rangle \) is a metric subspace of \( \langle X, d \rangle \), is often referred to as \( \langle Y, d \rangle \). If \( Y \neq X \), then it is a proper subspace.

**Theorem 1.2.** (Cartesian Product) Let \( \langle X, d_x \rangle \) and \( \langle Y, d_y \rangle \) be two metric spaces. Let \( Z = X \times Y \). Given \( 1 \leq p < \infty \), define \( \forall z_1, z_2 \in Z, z_1 = (x_1, y_1), z_2 = (x_2, y_2) \),

\[
d_p(z_1, z_2) = \left( d_x(x_1, x_2)^p + d_y(y_1, y_2)^p \right)^{1/p}
\]  

(7)

\[
d_\infty(z_1, z_2) = \max\{d_x(x_1, x_2), d_y(y_1, y_2)\}
\]  

(8)

Then, \( \{Z, d_p\} \) and \( \{Z, d_\infty\} \) are metric (product) spaces.

1.2. Open and closed sets

Let \( \langle X, d_x \rangle \) be any arbitrary metric space.

**Definition 1.3.** Let \( x_0 \in X \), and \( r > 0 \). An open ball (sphere, neighborhood), denoted by \( B(x_0, r) \), of radius \( r \) centered at \( x_0 \) is defined as

\[
B(x_0, r) = \{x \in X \mid d(x_0, x) < r\}
\]  

(9)

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1If \( S \) is a set of real numbers bounded above, then there is a smallest real number \( y \) such that \( x \leq y \) for all \( x \in S \). The number \( y \) is the least upper bound, called supremum of \( S \), denoted by \( \sup_{x \in S} x \). If \( S \) is not bounded above, \( \sup_{x \in S} x = \infty \). Similarly, the greatest lower bound of \( S \) is called infimum, denoted by \( \inf_{x \in S} x \).
For different metric, the open ball is different. Consider in $\mathbb{R}^2$, the open ball defined by $d_1$, $d_2$ and $d_\infty$ defined in (1), (2), and (4).

**Definition 1.4.** Let $Y$ be a subset of $X$. A point $x \in X$ is called an **interior point** of $Y$ if $\exists r > 0, \exists B(x, r) \subset Y$. The collection of all interior points of $Y$ is called the **interior** of $Y$, denoted by $Y^\circ$. A point $x \in X$ is called an **exterior point** of $Y$ if $x$ is an interior point of the complement of $Y$.

**Definition 1.5.** A set $Y$ is **open** in $X$ if $Y = Y^\circ$.

Examples of open set:

[1] In metric space $\langle \mathbb{R}, d_1 \rangle$, any arbitrary open interval $(a, b)$, where $a < b$, is open.

[2] Consider metric space $\langle C[a, b], d_\infty \rangle$. Let $\lambda > 0$ be an arbitrary finite number. The set of continuous function satisfying $|x(t)| < \lambda, \forall t \in [a, b]$ is an open set in $C[a, b]$.

Note that a set may have empty interior. For example, a point or a line in $\langle \mathbb{R}^2, d_2 \rangle$.

**Theorem 1.3.** Let $\langle X, d \rangle$ be a metric space.

(i) $X$ and $\emptyset$ are open sets.

(ii) If $\{Y_\alpha\}_{\alpha \in A}$ is an arbitrary family of open subsets of $X$, then $\bigcup_{\alpha \in A} Y_\alpha$ is an open set.

(iii) The intersection of finite number of open sets of $X$ is open.

Note that (iii) does not hold for infinite number of open sets. Counter example: In $\langle \mathbb{R}, d_1 \rangle$, let $S_n = (0, 1 + 1/n) \cap \bigcap_{n=1}^\infty S_n = (0, 1]$ which is not open.

**Definition 1.6.** Let $\langle X, d \rangle$ be a metric space. The **topology** of $X$ determined by $d$ is defined by the family of all open subsets of $X$.

**Definition 1.7.** Let $Y$ be a subset of metric space $\langle X, d \rangle$. A point $x \in X$ is called an **closure point** of $Y$ if $\forall \varepsilon > 0, \exists y \in Y \cap B(x, \varepsilon)$. The set of all closure points of $Y$ is called the **closure** of $Y$, denoted by $\bar{Y}$. A closure point $x$ is an **isolated point** if $\exists \varepsilon > 0, \exists B(x, \varepsilon) \cap Y = \{x\}$. A closure point $x$ is a **limit point** (or point of accumulation) of $Y$ if $\forall \varepsilon > 0, B(x, \varepsilon)$ contains infinite number of points of $Y$.

**Theorem 1.4.** Let $Y$ be a subset of metric space $\langle X, d \rangle$. If $x$ is a closure point of $Y$, then $x$ is either a limit point or an isolated point.

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A topology $\tau$ on a set $X$ is a collection of open subsets of $X$ satisfying: (a) $X, \emptyset \in \tau$, (b) $\tau$ is closed under arbitrary unions. (c) $\tau$ is closed under finite intersections. The set $X$ together with its topology $\tau$, $\langle X, \tau \rangle$, is called a topological space.
A closure point \( x \) of subset \( Y \) can be (a) an isolated point in \( Y \), (b) a limit point in \( Y \), or (c) a limit point that is not in \( Y \).

**Definition 1.8.** A set \( Y \) is closed in \( X \) if \( Y = \bar{Y} \).

**Theorem 1.5.** Let \( (X, d) \) be a metric space. A subset \( Y \) of \( X \) is closed iff its complement is open.

Examples of closed set:

1. In metric space \( (\mathbb{R}, d_1) \), any arbitrary closed interval \([a, b]\), where \( a < b \), is closed.
2. \( X = (-2, -1) \cup (1, 2) \) with the usual metric \( d_1 \). \( Y = (-2, -1) \) and \( Z = (1, 2) \) are both open subsets of \( X \). \( Y^c = Z \) and \( Z^c = Y \), so \( Y \) and \( Z \) are closed subsets of \( X \).

**Theorem 1.6.** Let \( (X, d) \) be a metric space.

(i) \( X \) and \( \emptyset \) are closed sets.

(ii) If \( \{Y_\alpha\}_{\alpha \in A} \) is an arbitrary family of closed subsets of \( X \), then \( \cap_{\alpha \in A} Y_\alpha \) is a closed set.

(iii) The union of finite number of closed sets of \( X \) is closed.

Again, (iii) does not hold for infinite number of closed sets. Counter example: In \( (\mathbb{R}, d_1) \), let \( S_n = [1/n, 1] \cup \bigcup_{n=1}^{\infty} S_n = (0, 1] \) which is not closed.

**Definition 1.9.** Let \( (X, d) \) be a metric space. A subset \( D \) is dense in \( X \) if every point of \( X \) is a closure point of \( D \). That is, \( \bar{D} = X \). A subset \( E \) is nowhere dense in \( X \) if \( \bar{E} \) contains no open set. The metric space \( (X, d) \) is separable if \( X \) contains a countable dense set.

Examples:

1. \( \mathbb{Q} \) is dense in \( (\mathbb{R}, d_1) \), \( \mathbb{Q} \) is countable, so \( (\mathbb{R}, d_1) \) is separable.
2. The set of polynomials with rational coefficients defined on \([a, b]\) is countable, and dense in \( (C[a, b], d_\infty) \), so \( (C[a, b], d_\infty) \) is separable.
3. \( \mathbb{N} \) is nowhere dense in \( (\mathbb{R}, d_1) \).
4. A line is nowhere dense in \( (\mathbb{R}^2, d_2) \).