1.5. **Sequential compactness**

Like closedness, compactness can also be characterized with convergent sequence, called sequential compactness. Both the topological definition and the sequential definition are useful in applications. We will show that the two definitions are equivalent.

**Definition 1.16.** Let $K$ be a subset of a metric space $(X, d)$. $K$ is **sequentially compact** if for any arbitrary sequence $\{x_n\}$ in $K$, there is a subsequence $\{x_{n_i}\}$ converge to an element $x \in K$.

**Theorem 1.14.** If $K$ is sequentially compact in metric space $(X, d)$, $K$ is closed.

A property equivalent to sequential compactness is Bolzano-Weierstrass property.

**Definition 1.17.** A metric space $(X, d)$ possesses the Bolzano-Weierstrass property if every infinite subset of $X$ has at least one limit point. A subset $Y$ of $X$ possesses the Bolzano-Weierstrass property if subspace $(Y, d)$ possesses the Bolzano-Weierstrass property.

**Theorem 1.15.** A subset $K$ a metric space $(X, d)$ is sequentially compact iff $K$ possesses the Bolzano-Weierstrass property.

Applying Theorem 1.15, the following theorem states that compactness implies sequential compactness.

**Theorem 1.16.** If a subset $K$ is compact in metric space $(X, d)$, then it is sequentially compact.

Next, we introduce the concept of totally boundedness, a condition stronger than boundedness.

**Definition 1.18.** A subset $K$ of a metric space $(X, d)$ is **totally bounded** if $\forall \varepsilon > 0$, there exists a finite subset $B_\varepsilon$ of $X$ such that $\forall x \in K$, $\exists y \in B_\varepsilon$, $\exists x \in B(y, \varepsilon)$. The set $B_\varepsilon$ is call an $\varepsilon$-net of $K$.

**Theorem 1.17.** If subset $K$ is totally bounded in metric space $(X, d)$, then it is bounded.

The converse of Theorem 1.17 is not true. For example, in the infinite-dimensional metric space $(l_2, d_2)$ defined in equations (11) and (12), the set $S = \{x \mid x \in l_2, \sum_{i=1}^{\infty} x_i^2 \leq 1\}$ is bounded, but it is not totally bounded. (Consider the set of points $E = \{e^1, e^2, \ldots\}$, where for any $i \in \mathbb{N}_+$, $e^i$ is the infinite-dimensional unit vector with value 1 in the $i$-th dimension and 0 in other dimensions. $E \subset S$, but $E$ can not be covered by any finite $\varepsilon$-net.)

**Theorem 1.18.** A subset $K$ a metric space $(X, d)$ is sequentially compact iff $K$ is totally bounded and complete.
To show the converse of Theorem 1.16, we need one more preliminary result.

**Lemma 1.19.** Let $K$ be a sequentially compact set in metric space $\langle X, d \rangle$. If $\{O_\alpha\}_{\alpha \in A}$ is an open cover of $K$, there exists an $\varepsilon > 0$ such that $\forall x \in K$, $\exists \alpha \in A$, $\exists B(x, \varepsilon) \subseteq O_\alpha$.

**Theorem 1.19.** If a subset $K$ is sequentially compact in metric space $\langle X, d \rangle$, then it is compact.